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Semi-Functional Partial Linear Quantile Regression Model with Randomly Censored Responses

Nengxiang Ling[1] · Jintao Yang[1] · Tonghui Yu[1] · Hui Ding[2] · Zhaoli Jia[1]

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Abstract
Censored data with functional predictors often emerge in many fields such as biology, neurosciences and so on. Many efforts on functional data analysis (FDA) have been made by statisticians to effectively handle such data. Apart from mean-based regression, quantile regression is also a frequently used technique to fit sample data. To combine the strengths of quantile regression and classical FDA models and to reveal the effect of the functional explanatory variable along with nonfunctional predictors on randomly censored responses, the focus of this paper is to investigate the semi-functional partial linear quantile regression model for data with right censored responses. An inverse-censoring-probability-weighted three-step estimation procedure is proposed to estimate parametric coefficients and the nonparametric regression operator in this model. Under some mild conditions, we also verify the asymptotic normality of estimators of regression coefficients and the convergence rate of the proposed estimator for the nonparametric component. A simulation study and a real data analysis are carried out to illustrate the finite sample performances of the estimators.

Keywords Functional data analysis · Quantile regression · Semi-functional partial linear model · Random censorship · Asymptotic properties

Mathematics Subject Classification 62G20 · 62G05

✉ Nengxiang Ling
hfut.lnx@163.com

1 School of Mathematics, Hefei University of Technology, Hefei 230009, People's Republic of China

2 School of Economics, Nanjing University of Finance and Economics, Nanjing 210023, People's Republic of China

1 Introduction

Functional data analysis (FDA) has become an important and very broad research field in modern statistics. More basic introductions and deep discussions on the FDA can be referred to [30] for the methods and the case studies, and also [29] for the parametric regression modeling, as well as [12] for the non-parametric FDA and its theoretical and pratical studies. The latest developments of FDA can be found in the monograph of [14] which provided not only the detailed descriptions of the testing algorithms and the performance by means of simulations studies but also the asymptotic theory of FDA. While [15] showed the theoretical foundations of FDA by introducing linear operators. Furthermore, some advances in nonparametric or semi-parametric statistical methods or models for FDA can be found in [1, 3, 25] among others.

Quantile regression (QR) is one of the most important statistical techniques which was initially developed to measure the relationships between the response and the explanatory variable besides mean-based regression methods, and has received considerable attention since the seminal work of [22]. In the case of FDA, or when some of explanatory variables are functional, some scholars have tried to combine the strengths of quantile regression with FDA, and to study the relative theories and applications.

For example, [7] developed a conditional quantile method for FDA. Their main purpose was to estimate conditional distribution function under a generalized functional regression framework when the explanatory variable takes its values in a functional space. [20] investigated the functional linear quantile regression model, and obtained the optimal convergence rate for the proposed estimators under suitable norms in a minimax sense. [27] considered the estimation of a functional partially quantile regression model, and provided asymptotical normality of the proposed estimator of the finite-dimensional parameter, and the convergence rate of the estimator of the infinite-dimensional slope function. [35] presented an estimation method for the partial functional linear quantile regression in the presence of both the functional and the non-functional predictors, and established some asymptotic properties for the proposed estimator under some mild conditions. [40] based on the partial least square basis method and selected partial quantile regression basis via maximizing the partial quantile covariance, which can make effective use of both information of covariates and responses in estimation. [8] adopted the kernel-based three-stage procedure to estimate the nonparametric regression operator and the partial linear coefficients in a semi-functional partial linear quantile regression model. [41] considered the composite quantile estimation for the partial functional linear regression model with errors forming a short-range dependent and strictly stationary linear process, and gave the large-sample properties of the proposed estimators and also displayed some applications in electricity consumption data. In addition, [31] proposed a functional single-index quantile regression model, where a generalized profiling method was employed to estimate the model.

It should be noted that all the mentioned works involved above are in the case that the samples are observed completely. However, in many practical occasions such as sampling survey, survival analysis, pharmaceutical tracing test and reliability test and so on, some pairs of observations may be incomplete such as missing responses at random (see [11, 24, 39]), or the responses being randomly censored [6, 16], or even the explanatory variables being functional data with partial observations [23, 38]. But it seems that the studies on functional quantile regression with incomplete observations are not adequate enough, and the research results are relatively few. Inspired by all the contributions above, in this paper, we focus on the estimations of the semi-functional partial linear quantile regression model with responses being randomly censored, and aim to establish some asymptotical properties of the proposed estimators. We also show the effectiveness of the method through some simulations and a real data analysis. In fact, our motivation comes from two aspects: one is from the real data analysis of the Alzheimer's Disease Neuroimaging Initiative(ADNI) database (https://adni.loni.usc.edu) in the setting of FDA, where the individual survival time with Alzheimer's Diseases may be censored, and the other is that many examples of the response subject to censorship and its related statistical inferences can be found in literature when all the explanatory variables are of finite dimensionality, one can refer to [5, 9, 28, 37] among others.

In what follows, for any quantile level $\tau \in (0, 1)$, we consider the semi-functional partial linear quantile regression model as follows:

$$\widetilde{Y} = \boldsymbol{Z}^T \boldsymbol{\beta}_\tau + m_\tau(\boldsymbol{\chi}) + \varepsilon_\tau, \tag{1.1}$$

where $\widetilde{Y}$ is the responses variable, $\boldsymbol{Z} = (Z_1, ..., Z_p)^T$ is a p-dimensional real-valued explanatory vector and $\boldsymbol{\beta}_\tau = (\beta_{1\tau}, ..., \beta_{p\tau})^T$ is an unknown p-dimensional coefficients vector belonging to a compact subset of $\mathbb{R}^p$, $\boldsymbol{\chi}$ is a functional explanatory variable that takes its value in a semi-metric space $\mathcal{F}$ with the associated semi-metric denoted by $d(\cdot, \cdot)$, $m_\tau(.)$ is an unspecified smooth functional operator from $\mathcal{F}$ to $\mathbb{R}$, ε_τ is a random error whose τ- th conditional quantile being zero given $(\boldsymbol{Z}, \boldsymbol{\chi})$. In the presence of randomly right censoring, the responses variable $\widetilde{Y}$ can be only observed as a pair of $Y = \min(\widetilde{Y}, C)$ and the censoring indicator $\Delta = I(\widetilde{Y} \leq C)$, where C is the censoring variable and $I(A)$ is an indicator function of a set A.

The rest of this paper is organized as follows. In Sect. 2, we propose the estimation procedure for the unknown parameter vector $\boldsymbol{\beta}_\tau$ and the unknown regression operator $m_\tau(.)$ based on incomplete dataset following model (1.1). Section 3 presents some necessary regularity conditions and the main theoretical results of this paper. To evaluate the finite-sample performances of the proposed estimators, some simulation studies and a real data analysis are carried out in Sects. 4 and 5, respectively to show the finite-sample performances of the proposed estimators. Then, some concluding remarks and potential future works are given in Sect. 6. Finally, the technical proofs of some lemmas and the main results are relegated to Sect. 7.

2 Methodology

Let $\{(Y_i, \Delta_i, Z_i, \chi_i), i = 1, ..., n\}$ be n *i.i.d* realizations of $(Y, \Delta, \mathbf{Z}, \boldsymbol{\chi})$, and satisfy model (2.1):

$$\tilde{Y}_i = \boldsymbol{Z}_i^T \boldsymbol{\beta}_\tau + m_\tau(\boldsymbol{\chi}_i) + \varepsilon_{i\tau}, \tag{2.1}$$

where $Y_i = \min\{\widetilde{Y}_i, C_i\}$, $\Delta_i = I(\widetilde{Y}_i \le C_i)$, $Z_i = (Z_{i1}, ..., Z_{ip})^T$ and χ_i are observed completely for $i = 1, 2, ..., n$. Throughout this paper, we assume that the censoring variables $\{C_i\}_{i=1}^n$ are *i.i.d* with common unknown survival function $G(t) = P\big(C > t\big)$, and also independent of $(\widetilde{Y}_i, Z_i, \chi_i)$. Our initial goal is to construct the estimators of the unknown parameters vector $\boldsymbol{\beta}_\tau$ and regression operator $m_\tau(\cdot)$ in model (2.1). Due to the existence of random censorship, we exploit the inverse-censoring-probability-weighted (ICPW) strategy and consider the following weighted objective function to estimate the parameters or operator in model (2.1):

$$\sum_{i=1}^{n} \frac{\Delta_i}{G(Y_i)} \rho_\tau\big(Y_i - Z_i^\top \beta_\tau - m_\tau(\chi_i)\big), \tag{2.2}$$

where $\rho_\tau(s) = s(\tau - I(s < 0))$ is a quantile loss function. Generally, $G(\cdot)$ is unknown in practice, and is usually estimated with [18] (K-M) method that can be expressed as

$$\widehat{G}_n(u) = \begin{cases} \prod_{i=1}^{n} \big(1 - \frac{1 - \Delta_{(i)}}{n - i + 1}\big)^{I_{[Y_{(i)} \le u]}}, & \text{if} \quad u < Y_{(n)}, \\ 0, \quad \text{Otherwise}, \end{cases} \tag{2.3}$$

where $Y_{(1)} \le Y_{(2)} \le ... \le Y_{(n)}$ are the order statistics of Y_i, and $\Delta_{(i)}$ is the indicator variable corresponding to $Y_{(i)}$. Thus, substituting $G(.)$ with $\widehat{G}_n = \widehat{G}_n(.)$ into the loss function (2.2) derives the objective function as follows:

$$Q_n(\beta_\tau, m_\tau(.), \widehat{G}_n) = \sum_{i=1}^{n} \frac{\Delta_i}{\widehat{G}_n(Y_i)} \rho_\tau\big(Y_i - Z_i^\top \beta_\tau - m_\tau(\chi_i)\big). \tag{2.4}$$

At first glance, the estimators for $(\beta_\tau, m_\tau(.))$ can be defined as the point of minima of the weighted loss function (2.4). But such way will result in inefficient estimation owing to mixture of linear and nonlinear components. In line with [4] and [8], we develop the following three-step estimation procedure for the semi-functional partial linear quantile regression model (2.1) in which responses may be randomly censored.

First, by minimizing the following local weighted quantile loss function, we obtain an initial estimators $\widetilde{\beta}_\tau$ and $\widetilde{a}_\tau(\chi)$ of β_τ and $m_\tau(\chi)$, respectively, as following:

$$(\widetilde{\beta}_\tau, \widetilde{a}_\tau(\chi)) = \arg \min_{\beta_\tau, a_\tau(\chi)} \sum_{i=1}^{n} \frac{\Delta_i}{\widehat{G}_n(Y_i)} \rho_\tau\big(Y_i - Z_i^\top \beta_\tau - a_\tau(\chi)\big) K_h(d(\chi_i, \chi)), \tag{2.5}$$

where $K_h(\cdot) = K(\cdot/h)$ and $K(\cdot)$ is a kernel function and $h = h_n > 0$ is a bandwidth with $h_n \to 0$ as $n \to \infty$. Second, for an initial estimator $\widetilde{a}_\tau(\chi)$, we have the final estimator of $\boldsymbol{\beta}_\tau$ by minimizing the following quantile loss function:

$$\widehat{\beta}_\tau = \arg\min_{\beta_\tau} \sum_{i=1}^{n} \frac{\Delta_i}{\widehat{G}_n(Y_i)} \rho_\tau\big(Y_i - Z_i^\top \beta_\tau - \widetilde{a}_\tau(\chi)\big). \tag{2.6}$$

Since nonparametric and parametric components have different rates of convergence in estimation, the second step will enhance the estimation efficiency regarding parameter β_τ compared to the initial estimators. Thirdly, with the ICPW estimate $\widehat{\beta}_\tau$, the final estimator of $m_\tau(\chi)$ is obtained by minimizing the following local weighted quantile loss function:

$$\widehat{a}_\tau(\chi) = \arg\min_{a_\tau(\chi)} \sum_{i=1}^{n} \frac{\Delta_i}{\widehat{G}_n(Y_i)} \rho_\tau\big(Y_i - Z_i^\top \widehat{\beta}_\tau - a_\tau(\chi)\big) K_h(d(\chi_i, \chi)). \tag{2.7}$$

3 Theoretical Results

3.1 Notations and Assumptions

In this subsection, we first give some additional notations. Specifically, let $B(\chi, h) = \{y : d(y, \chi) < h\}$ denote an open ball with center χ and radius h, and $S_{\mathcal{F}}$ be a compact subset of $\mathcal{F}$. Furthermore, Similar to [10], let $N_\epsilon(S_{\mathcal{F}})$ be the minimal number of open balls in $\mathcal{F}$ with centers χ_1, ..., $\chi_{N_\epsilon(S_{\mathcal{F}})}$ and radius ϵ to cover $S_{\mathcal{F}}$, and $\psi_{S_{\mathcal{F}}}(\epsilon) = \log(N_\epsilon(S_{\mathcal{F}}))$ be the Kolmogrov's ϵ- entropy of $S_{\mathcal{F}}$. The covering is called the $\varepsilon-$ net of $S_{\mathcal{F}}$. For convenience, let $f_\tau(\cdot|\mathbf{Z}, \boldsymbol{\chi})$ and $F_\tau(\cdot|\mathbf{Z}, \boldsymbol{\chi})$ denote the conditional density function and the conditional cumulative distribution function of the error ε_τ given $(\mathbf{Z}, \boldsymbol{\chi})$, respectively. Denoting $H_\tau(\chi) =: \mathbb{E}\left\{f_\tau(0|\mathbf{Z}, \boldsymbol{\chi})(\mathbf{1}, \mathbf{Z}^\top)^\top(\mathbf{1}, \mathbf{Z}^\top)|\boldsymbol{\chi} = \chi\right\}$, $A_\tau(\mathbf{Z}, \chi) =: \mathbb{E}\left\{f_\tau(0|\mathbf{Z}, \boldsymbol{\chi})\mathbf{Z}(\mathbf{1}, \mathbf{0}^\top)|\boldsymbol{\chi} = \chi\right\} H_\tau(\chi)^{-1}(1, \mathbf{Z}^\top)$.

In what follows, in order to show the main results of this paper, we need to present some assumptions. Throughout this paper, let $c, c_1, c_2, ...$ be some positive constants not depending on n which may take different values in each appearance.

(A1) There exist constants $c_1 > 0$ and $\alpha > 0$ such that for any $u, v \in S_{\mathcal{F}}$, $|m_\tau(u) - m_\tau(v)| \le c_1 d(u, v)^\alpha$.

(A2) There exist constants $c_2 > 0$ and $c_3 > 0$ and a function $\phi(h)$ on $(0, \infty)$ such that $0 < c_2\phi(h) \le P\{\boldsymbol{\chi} \in B(\chi, h)\} \le c_3\phi(h)$ for any $\chi \in S_{\mathscr{F}}$.

(A3) (i) $K(\cdot)$ is a bounded nonnegative function with support $[0, 1]$ and satisfies a Lipschitz condition on $[0, 1)$. (ii) If $K(1) = 0$, $K(\cdot)$ satisfies an additional assumption: its derivative $K^{'}(\cdot)$ exists on $[0, 1]$ with $-\infty < c_4 \le K^{'}(u) \le c_5 < 0$ for $c_4 > 0$ and $c_5 > 0$.

(A4) (i) $\exists\, c_5 > 0$ and $\exists \varepsilon_0 > 0$ such that $\forall 0 < \varepsilon < \varepsilon_0$, $\phi^{'}(\varepsilon) < c_5$. (ii) If $K(1) = 0$, then $\exists c_6 > 0$ and $\exists \varepsilon_0 > 0$ such that $\forall 0 < \varepsilon < \varepsilon_0$, $\int_0^\varepsilon \phi(u)du > c_6\varepsilon\phi(\varepsilon)$.

(A5) Kolmogorov's ε-entropy of $S_{\mathcal{F}}$ satisfies:

$$(i) \quad \frac{\log^2 n}{n\phi(h)} < \psi_{S_{\mathcal{F}}}\left(\frac{\log n}{n}\right) < \frac{n\phi(h)}{\log n}$$

and

$$(ii)\quad \sum_{n=1}^{\infty}\exp\{(1-v)\psi_{S_{\mathcal{F}}}(\frac{\log n}{n})\}<\infty, \quad for\ \ some\ \ v>1.$$

(A6) The scalar covariate $\mathbf{Z}$ is uniformly bounded and $\mathbb{E}[f_\tau(0|\mathbf{Z},\boldsymbol{\chi})\mathbf{Z}\mathbf{Z}^\top|\boldsymbol{\chi}=\chi]$ is a finite and nondegenerate matrix.

(A7) $f_\tau(.|\mathbf{Z},\boldsymbol{\chi})$ is bounded away from zero and has a continuous and uniformly bounded derivative.

(A8) There exists a maximum follow-up denoted by L and a constant $\nu_0>0$ such that $P\{t\leq\widetilde{Y}\leq C\}\geq\nu_0>0$ for any $t\in[0,L]$.

(A9) $H_\tau(\chi)$ is continuous and nonsingular on $S_{\mathcal{F}}$.

Comments on the assumptions: Assumptions (A1)–(A4) are quite usual conditions in the context of nonparametric functional regression modeling, one can refer to [12] for details. Assumption (A5) shows the topological considerations by restrictions on Kolmogrov's $\varepsilon-$ entropy of $S_{\mathcal{F}}$, which had been adopted by [10] and [8] to get the uniform convergence rates. Assumptions (A6),(A7) and (A9) are commonly used for quantile regression model, see [8, 17] and [4]. Assumption (A8) is an usual condition in survival analysis model with responses subject to random censorship, which had been adopted by [33, 34] and [9] among others.

3.2 Main Results

In this subsection, we present the main results of this paper as follows.

Theorem 3.1 *Under the assumptions* (A1)–(A9), *if in addition,* $\dfrac{\psi_{S_{\mathcal{F}}}\left(\dfrac{\log\ n}{n}\right)}{n\phi^2(h)}\longrightarrow 0$ *as* $n\longrightarrow\infty$, *then we have*

$$\sqrt{n}(\widehat{\boldsymbol{\beta}}_\tau-\boldsymbol{\beta}_\tau)\xrightarrow{d}N(0,\Sigma_{\mathbf{1}\tau}^{-\mathbf{1}}\Sigma_{\mathbf{2}\tau}\Sigma_{\mathbf{1}\tau}^{-\mathbf{1}}),\tag{3.1}$$

where $\boldsymbol{\Sigma}_{1\tau}=\mathbb{E}\left(f(0|\mathbf{Z},\boldsymbol{\chi})\mathbf{Z}\mathbf{Z}^\top\right)$, $\boldsymbol{\Sigma}_{2\tau}=\mathbb{E}\left[((\mathbf{Z}-A_\tau(\chi,z))r)^{\otimes 2}\right]$

$$+\,\mathbb{E}\left[\left(\int_0^L(\mathbf{Z}-A_\tau(\chi,z))r-\frac{\sum_{j=1}^n\dfrac{\Delta_j}{G(Y_j)}\dfrac{1}{n}(I(Y_j\geq u)(\mathbf{Z_j}-\mathbf{A}_\tau(\chi_\mathbf{j},\mathbf{z_j})))\mathbf{r_j}}{S(u)}\right)^{\otimes 2}\right.$$

$$\left.I(Y\geq u)\frac{\lambda(u)}{G^2(u)}du\right]$$

with $\lambda(u)=\dfrac{d\Lambda_c(u)}{du}$ *and* $S(u)=P(\widetilde{Y}\geq u)$. *Here* $\Lambda_c(u)$ *is the cumulative hazard function of the censoring variable* C, *and the symbol* $\boldsymbol{a}^{\otimes 2}$ *denotes the outer product of a vector* $\boldsymbol{a}$ *and itself,* $r=I(\varepsilon_\tau\leq 0)-\tau$ *and* $r_j=I(\varepsilon_{j\tau}\leq 0)-\tau$ *for* $j=1,2,...,n$.

Theorem 3.2 *Under the assumptions* (A1)-(A8), *if in addition*, $nh^{2\alpha} \longrightarrow \infty$, *then we have, as* $n \longrightarrow \infty$,

$$\sup_{x \in S_{\mathcal{F}}} |\widehat{m}_\tau(\chi) - m_\tau(\chi)| = O_p\left(h^\alpha + \sqrt{\frac{\psi_{S_{\mathcal{F}}}\left(\frac{\log n}{n}\right)}{n\phi(h)}} \right). \tag{3.2}$$

Comments on the main results. The results extend that of [4] and [9] in the case of non-functional explanatory variable to what the functional explanatory variable is involved. Meanwhile, both Theorems 3.1 and 3.2 extend [8] to the case where the responses are randomly censored.

4 Simulation Studies

This section is devoted to illustrating the finite sample performance of the proposed methodology. For that purpose, let us consider the following model:

$$\widetilde{Y}_i = Z_{1i}\beta_{1\tau} + Z_{2i}\beta_{2\tau} + m_\tau(\chi_i) + \varepsilon_i(\tau), \qquad i = 1, 2, ..., n.$$

where $\{Z_{1i}\}_{i=1}^n$ and $\{Z_{2i}\}_{i=1}^n$ are independent with the standard normal distribution $N(0, 1)$. The functional data sets are generated by the curves $\chi_i(t) = a_i(t - 0.5)^2 + b_i \quad (t \in [0, 1], \quad i = 1, 2, ...n)$ which had also been adopted by [2] as well as [8]. Here $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ are i.i.d. with uniform distribution $U(0, 1)$ and $U(-0.5, 0.5)$, respectively. The simulated curves are shown by Fig. 1. On the other hand, similar to [12], we choose the semi-metric which is defined by $d(\chi_i, \chi_j) = \sqrt{\int_0^1 \left(\chi_i(t) - \chi_j(t)\right)^2 dt}$, $(i, j = 1, 2, ..., n)$, and the kernel function $K(u) = \frac{3}{4}(1 - u^2)I_{[0,1]}(u)$. Meanwhile, the smoothing parameter h is selected by the multiple experiments on the mean square loss criterion. Denote $m(\chi_i) = \exp(-8\,g(\chi_i)) - \exp(-12\,g(\chi_i))$, where $g(\chi_i) = \text{sign}(\chi_i^{'}(1) - \chi_i^{'}(0))\sqrt{3\int_0^1 (\chi_i^{'}(t))^2 dt}$ with $\text{sign}(x) = 1$ if $x > 0$ and $\text{sign}(x) = -1$ if $x < 0$ as well as $\text{sign}(x) = 0$ if $x = 0$ and $\varepsilon_i(\tau) = \varepsilon_i - F^{-1}(\tau)$ with $F(.)$ being the common CDF of ε_i for $i = 1, 2, ...n$. We take the coefficients of numerical covariates as $\beta_{1\tau} = -1$ and $\beta_{2\tau} = 2$ for quantile level τ, respectively, and then consider the two different distribution for random error as follows:
Case (I): $\{\varepsilon_i\}_{i=1}^n$ is i.i.d. with normal distribution $N(0, 0.25)$.
Case (II): $\{\varepsilon_i\}_{i=1}^n$ is i.i.d. with a Cauchy distribution for scale being 0.2.

Similar to [13], the censoring variables $\{C_i\}_{i=1}^n$ are generated by the uniform distribution $U(0, c)$, where constant c controls the censoring proportion (CP).

In what follows, to evaluate the performance of the proposed method, we adopt the following quantities: the average bias (Abias) and the standard deviation (SD) of the

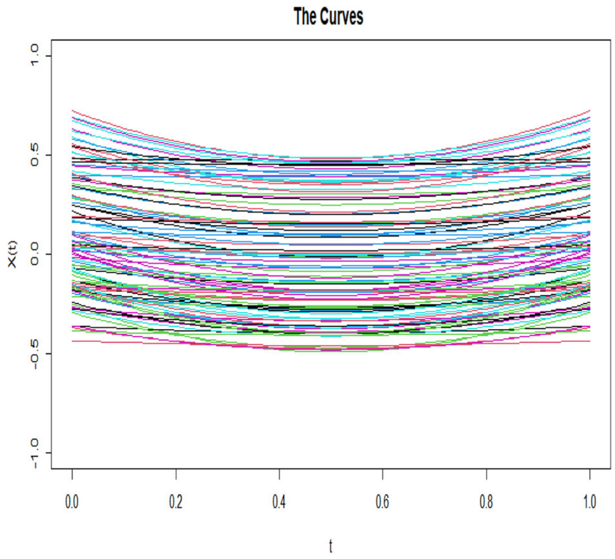

Fig. 1 The simulated sample curves $\{\chi_i(t)\}_{i=1}^n$, $t \in [0, 1]$.

estimators for the parametric components and the sample size n, which is defined as

$$Abias(\widehat{\beta}_{s\tau}) = \frac{1}{N}\sum_{j=1}^{N}\left(\widehat{\beta}_{s\tau j} - \beta_{s\tau}\right) \quad and \qquad SD(\widehat{\beta}_{s\tau}) = \left\{\frac{1}{N}\sum_{j=1}^{N}(\widehat{\beta}_{s\tau j} - \beta_{s\tau})^2\right\}^{\frac{1}{2}},$$

for $s = 1, 2$, as well as the average bias (Abias) and the root of average squared errors (RASE) of the estimators for the nonparametric components, which is also defined as

$$Abias(\widehat{m}) = \frac{1}{N}\sum_{j=1}^{N}\left\{\frac{1}{n}\sum_{i=1}^{n}\left(\widehat{m}_{\tau j}(\chi_i) - m_{\tau j}(\chi_i)\right)\right\}$$

and

$$RASE(\widehat{m}) = \frac{1}{N}\sum_{j=1}^{N}\left\{\frac{1}{n}\sum_{i=1}^{n}[\widehat{m}_{\tau j}(\chi_i) - m_{\tau j}(\chi_i)]^2\right\}^{\frac{1}{2}},$$

where χ_i, $(i = 1, 2, ..., n)$ are generated to evaluate the effectiveness of the model.

Based on $N = 1000$ simulation runs, we show the estimation results of the proposed estimators for $\boldsymbol{\beta}_\tau$ and $m_\tau(.)$ with quantile leavel $\tau = 0.25, 0.50, 0.75$ and the different CP of 20%, 30% and 50% as well as different sample sizes $n = 100, 200, 300$ under Cases (I) and (II) in Tables 1–4. These tables report the similar patterns, that is, averaged biases of estimators are satisfactorily small, and estimation precision in terms of RASE

Table 1 Abias and RASEs or SD (in parentheses) of $\widehat{m}_\tau$ and $\widehat{\beta}_{s\tau}$ under different quantile level for error distribution $N(0, 0.25)$ with the CP of 20%

n	τ	$\widehat{m}_\tau(\chi)$	$\widehat{\beta}_{1\tau}$	$\widehat{\beta}_{2\tau}$
100	0.25	0.0239(0.0057)	−0.0019(0.0226)	−0.0040(0.0262)
	0.50	0.0081(0.0028)	0.0025(0.0205)	−0.0037(0.0231)
	0.75	0.0029(0.0007)	0.0029(0.0227)	−0.0051(0.0257)
200	0.25	0.0237(0.0040)	−0.0006(0.0155)	0.0012(0.0164)
	0.50	0.0077(0.0019)	0.0001(0.0142)	−0.0016(0.0158)
	0.75	0.0026(0.0004)	0.0005(0.0151)	−0.0020(0.0176)
300	0.25	0.0237(0.0032)	0.0009(0.0121)	−0.0010(0.0134)
	0.50	0.0077(0.0015)	0.0018(0.0113)	−0.0022(0.0124)
	0.75	0.0025(0.0002)	0.0012(0.0127)	−0.0027(0.0143)

Table 2 Abias and RASEs or SD (in parentheses) of $\widehat{m}_\tau$ and $\widehat{\beta}_{s\tau}$ under different quantile level for error distribution $N(0, 0.25)$ with the CP of 30%

n	τ	$\widehat{m}_\tau(\chi)$	$\widehat{\beta}_{1\tau}$	$\widehat{\beta}_{2\tau}$
100	0.25	0.0240(0.0061)	−0.0017(0.0226)	0.0053(0.0264)
	0.50	0.0083(0.0031)	0.0026(0.0209)	−0.0042(0.0232)
	0.75	0.0030(0.0008)	0.0020(0.0226)	−0.0046(0.0260)
200	0.25	0.0240(0.0041)	−0.0003(0.0152)	−0.0011(0.0180)
	0.50	0.0079(0.0021)	0.0009(0.0140)	−0.0034(0.0163)
	0.75	0.0027(0.0004)	0.0014(0.0156)	−0.0028(0.0182)
300	0.25	0.0240(0.0034)	0.0007(0.0120)	−0.0019(0.0136)
	0.50	0.0078(0.0016)	0.0021(0.0116)	−0.0031(0.0129)
	0.75	0.0026(0.0003)	0.0020(0.0126)	−0.0033(0.0145)

of $\widehat{m}_\tau(\chi)$ and SD of $\widehat{\beta}_{1\tau}$ and $\widehat{\beta}_{2\tau}$ decreases as sample size n increases. By Tables 1–3, we can find that as CP increases from 20% to 50%, Abias and SD of $\widehat{\beta}_{1\tau}$ and $\widehat{\beta}_{2\tau}$ as well as Abias and RASE for $\widehat{m}_\tau(\chi)$ all get slightly worse, which is allowable because of lack of complete information and less valid data used in estimations for data with higher censoring rates. Estimation results under Case II and CP of 50% are reported in Table 4, while results under same set-up with lower censoring rate has similar tendency and thus can be omitted. Under Case II, the proposed estimators regarding the nonparametric component achieves the better performances at $\tau = 0.75$ in the light of Abias and RASE.

On the other hand, in order to show the asymptotic normal of the parameter estimators, the quantile-quantile (QQ) plots of the estimators are also shown in Fig. 2 and Fig. 3, respectively under the different error distribution and a fixed CP such as 30% for $\tau = 0.5$. We can also find that the parameter estimators are asymptotically normal distribution.

Table 3 Abias and RASEs or SD (in parentheses) of $\widehat{m}_\tau$ and $\widehat{\beta}_{s\tau}$ under different quantile level for error distribution $N(0, 0.25)$ with the CP of 50%

n	τ	$\widehat{m}_\tau(\chi)$	$\widehat{\beta}_{1\tau}$	$\widehat{\beta}_{2\tau}$
100	0.25	0.0240(0.0087)	−0.0057(0.0297)	0.0112(0.0332)
	0.50	0.0096(0.0048)	0.0038(0.0234)	−0.0072(0.0287)
	0.75	0.0035(0.0016)	0.0040(0.0254)	−0.0075(0.0331)
200	0.25	0.0251(0.0067)	−0.0001(0.0189)	0.0006(0.0229)
	0.50	0.0094(0.0036)	0.0048(0.0172)	−0.0086(0.0218)
	0.75	0.0031(0.0009)	0.0051(0.0195)	−0.0109(0.0252)
300	0.25	0.0254(0.0055)	0.0014(0.0147)	−0.0035(0.0181)
	0.50	0.0091(0.00284)	0.0039(0.0136)	−0.0077(0.0173)
	0.75	0.0028(0.0005)	0.0041(0.0141)	−0.0080(0.0190)

Table 4 Abias and RASEs or SD (in parentheses) of $\widehat{m}_\tau$ and $\widehat{\beta}_{s\tau}$ under different quantile level for a Cauchy error distribution (scale is 0.2) with the CP of 50%

n	τ	$\widehat{m}_\tau(\chi)$	$\widehat{\beta}_{1\tau}$	$\widehat{\beta}_{2\tau}$
100	0.25	0.1412(0.1836)	0.0386(0.0819)	−0.0810(0.1091)
	0.50	0.0285(0.0189)	0.0200(0.0389)	−0.0407(0.0510)
	0.75	0.0082(0.0074)	0.0262(0.0494)	−0.0458(0.0717)
200	0.25	0.0993(0.0532)	0.0311(0.0430)	−0.0624(0.0575)
	0.50	0.0255(0.0096)	0.0185(0.0235)	−0.0353(0.0318)
	0.75	0.0060(0.0037)	0.0235(0.0317)	−0.0462(0.0460)
300	0.25	0.0950(0.0379)	0.0300(0.0335)	−0.0598(0.0438)
	0.50	0.0246(0.0075)	0.0161(0.0187)	−0.0033(0.0263)
	0.75	0.0055(0.0029)	0.0228(0.0254)	−0.0447(0.0372)

5 A Real Data Analysis

In this section, we use a real data set to illustrate the proposed estimation procedures for the model. The data set can be shown at the Alzheimer's Disease Neuroimaging Initiative(ADNI) database (https://adni.loni.usc.edu). The detailed data analysis, including preprocessing of hippocampal image data and summary of demographic information summary can be found in [19], which is also adopted by [32]. Here, we also consider the clinical and imaging measures of 373 Mild Cognitive Impairment (MCI) individuals in ADNI. Among the 373 MCI individuals, 161 MCI individuals progressed to AD before the study completed, and the remaining 212 MCI individuals did not convert to AD prior to study end. Thus, the time of conversion from MCI to AD can be treated as time-to-event data and the censoring proportion of the data set is 57% (212/373).

Precisely, the scalar covariate that we choose includes gender (1=Male; 0=Female), handedness (1=Right; 0=Left), marital status (1=married; 0=widowed, divorced or never married), education length, retirement (1=Yes; 0=No), age, and the ADAS-Cog

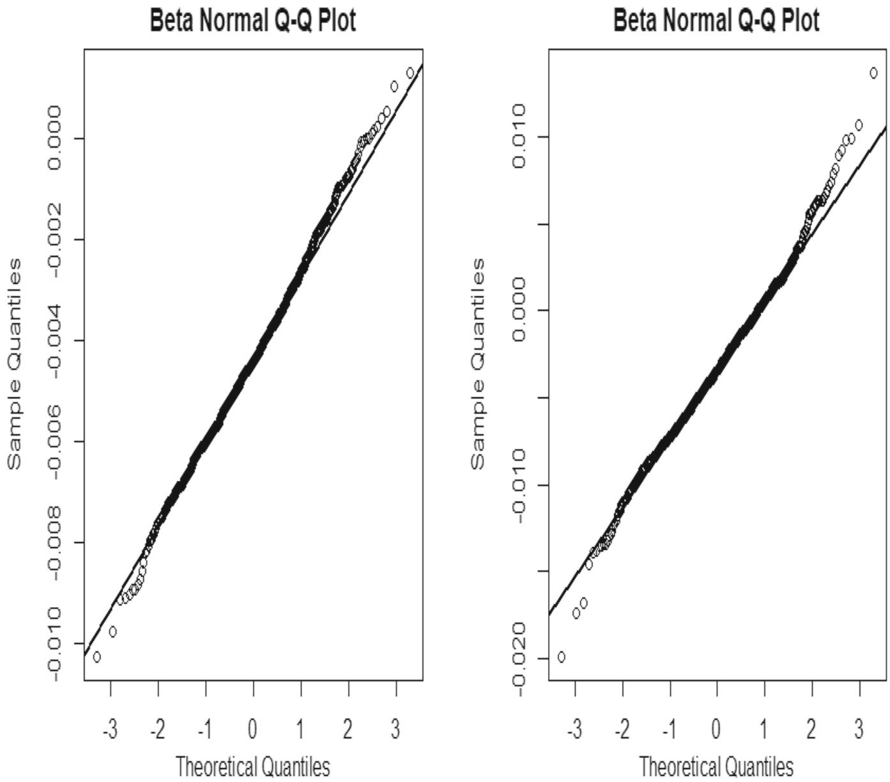

Fig. 2 QQ plots of the estimators $\widehat{\beta}_{1\tau}$ (left) and $\widehat{\beta}_{2\tau}$ (right) for error distribution $N(0, 0.25)$ with $\tau = 0.50$ and the CP of 30%

score. For the functional predictors, we use hippocampal radial distances of 30,000 surface points on the left and right hippocampus surfaces. The radial distance is defined as the distance between the medial core of the hippocampus and the corresponding vertex, and it is a summary statistic of the hippocampal shape and size. We consider the following the semi-functional partial linear quantile regression model to fit the data:

$$\begin{aligned}\widetilde{Y}_i &= Gender_i \times \beta_{\tau 1} + Handedness_i \times \beta_{\tau 2} + MS_i \times \beta_{\tau 3} \\ &\quad + Education_i \times \beta_{\tau 4} + Retirment_i \times \beta_{\tau 5} \\ &\quad + Age_i \times \beta_{\tau 6} + ADAS_i \times \beta_{\tau 7} + m_\tau(\chi_i) + \varepsilon_{\tau i},\end{aligned} \tag{5.1}$$

where $\widetilde{Y}$ is the log of the survival time and is randomly censored. Our aim is to estimate the model (5.1) between the scalar covariates mentioned above as well as the functional covariates for the time of MCI to AD transformation. Also, we choose the semi-metric and the kernel function which is the same as that in the case of simulation studies. The bandwidth involved in the kernel function are chosen by minimizing the cross-validation criterion as Fig. 4 shows. The estimation results of the scalar covariance at

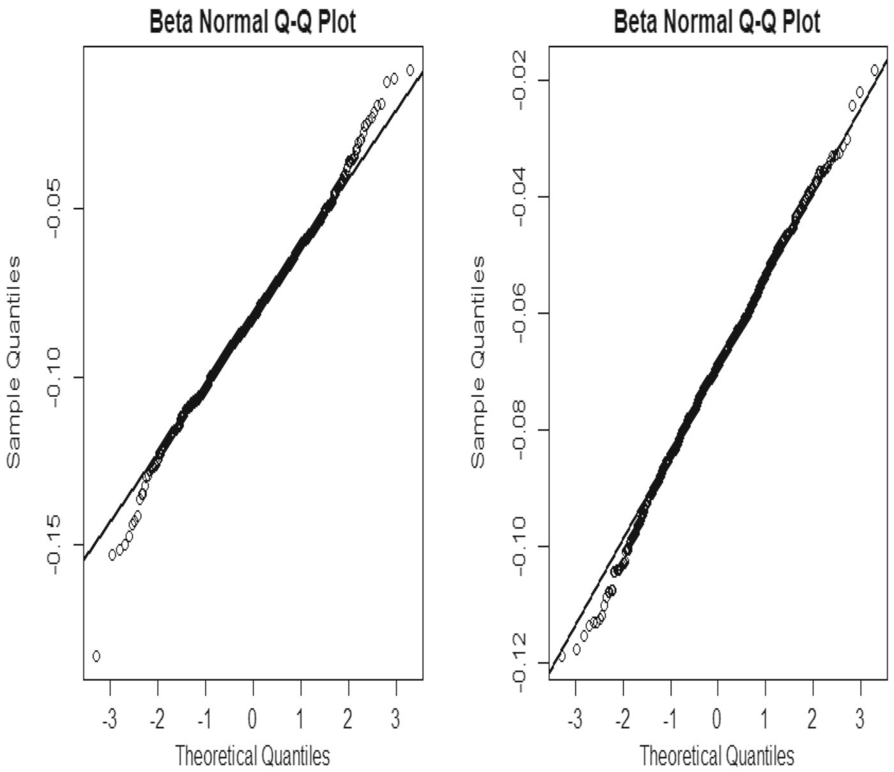

Fig. 3 QQ plots of the estimators $\widehat{\beta}_{1\tau}$ (left) and $\widehat{\beta}_{2\tau}$ (right) for a Cauchy error distribution (scale is 0.2) with $\tau = 0.50$ and the CP of 30%

different quantile levels are reported in Table 5. By Table 5, It can be found that the age covariate is always significant across all quantile levels, the ADAS-Cog score is significant over low quantile levels, and marital status has important effects on the time of conversion from MCI to AD over the middle range of quantile levels.

6 Conclusion

In this paper, we develop a new estimation procedure for the semi-functional partial linear quantile regression model in the case of randomly censored responses. Then some asymptotic properties of the estimators for the parameter coefficients and the non-parameter regression operator of the model are obtained respectively under some mild conditions. Both the simulation studies and the real data analysis shows that the proposed procedure is effective for this model.

On the other hand, it will be of interest to consider an extension from an i.i.d. sample to that of dependence such as functional time series data in the case of the randomly censored responses which requires some nontrivial mathematics techniques. Meanwhile, the variable selection for such estimating producer based on incomplete

Table 5 Parameter estimation and SD (in parentheses) of the model (5.1) for the ADNI data under different quantile level

τ	Gender	Handedness	MS	Education	Retirment	Age	ADAS
0.1	0.022(0.108)	0.946(0.188)**	0.451(0.131)**	−0.036(0.017)**	0.104(0.113)	0.029(0.005)**	−0.037(0.015)**
0.2	0.017(0.052)	0.069(0.09)	−0.036(0.063)	0.04(0.008)**	−0.416(0.054)**	0.029(0.002)**	−0.025(0.007)**
0.3	−0.2(0.1)**	0.353(0.173)**	0.08(0.121)	0.027(0.015)*	−0.272(0.105)**	0.03(0.005)**	−0.042(0.014)**
0.4	−0.166(0.17)	0.708(0.295)**	0.353(0.206)*	0.07(0.026)**	−0.208(0.178)	0.036(0.008)**	−0.032(0.023)
0.5	−0.485(0.103)**	0.455(0.178)**	0.456(0.125)**	0.069(0.016)**	−0.275(0.108)**	0.055(0.005)**	−0.011(0.014)
0.6	−0.686(0.105)**	0.071(0.183)	0.41(0.128)**	0.056(0.016)**	−0.495(0.111)**	0.05(0.005)**	−0.004(0.015)
0.7	−0.207(0.173)	0.41(0.3)	0.251(0.21)	0.049(0.027)*	−0.225(0.181)	0.03(0.008)**	0.006(0.024)
0.8	−0.188(0.279)	0.012(0.485)	0.209(0.34)	0.015(0.043)	−0.246(0.293)	0.027(0.013)**	−0.002(0.039)
0.9	−0.362(0.09)**	0.22(0.156)	0.052(0.109)	0.032(0.014)**	−0.043(0.094)	0.023(0.004)**	−0.013(0.012)

* and ** correspond to 0.1 and 0.05 significance level, respectively.

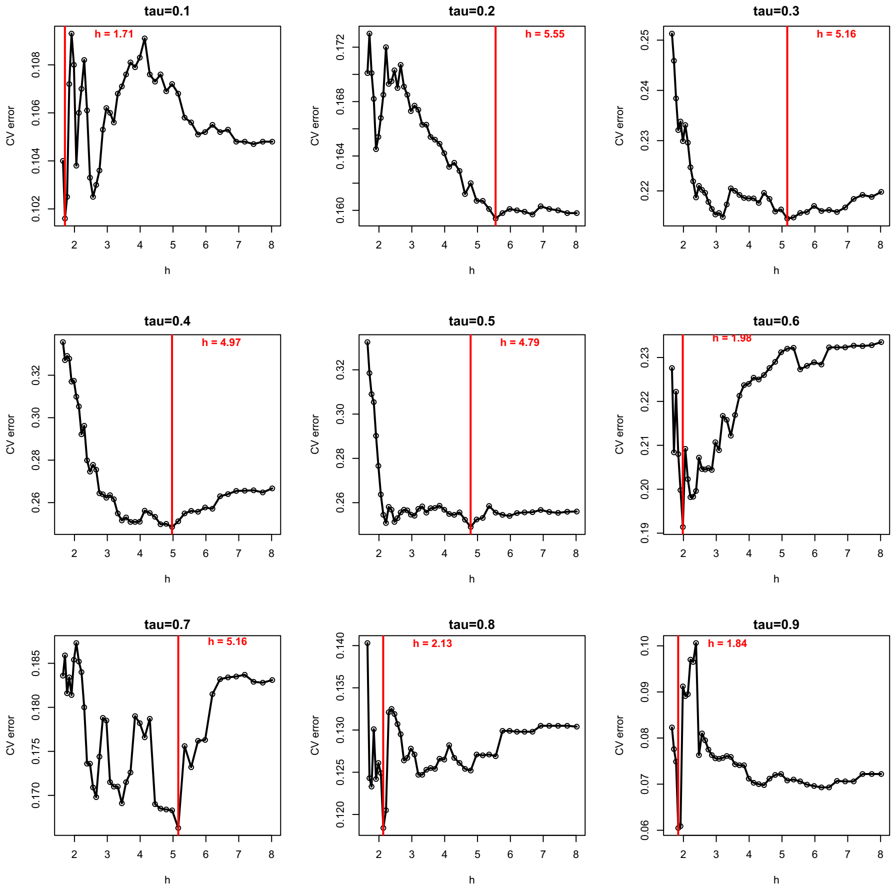

Fig. 4 Cross-validation errors with different bandwidths and quantile level at $\tau = 0.1, 0.2, ..., 0.9$ under the ADNI dataset

observation data may have more challenge which needs further investigation. Furthermore, just as one referee comments, the model investigated in this paper can be also extended to the generalized semi-functional partial linear quantile regression model with responses under random censorship. In addition, the censoring variable is conditionally independent of the response variable given some covariates is also interesting. All that goes beyond the scope of the present paper, and will be our next work in future.

7 Proofs

In this section, we will give some lemmas and their proofs which is helpful to prove the main results of this paper.

Lemma 7.1 *Under the assumptions* (A1)− (A5) *and* (A8), *for any random variables* B_i *with* $|B_i| \leq M < \infty$,$i = 1, 2, ...n$, *we have that*

$$\sup_{\chi \in S_{\mathcal{F}}} \left| \frac{1}{n\phi(h)]} \sum_{i=1}^{n} \left[K_h\left(d\left(\chi_i, \chi\right)\right) \frac{\Delta_i B_i}{G(Y_i)} - \mathbb{E} K_h\left(\chi_i, \chi\right) \frac{\Delta_i B_i}{G(Y_i)} \right] \right|$$
$$= O_p\left(\sqrt{\frac{\psi_{S_{\mathcal{F}}}\left(\frac{\log n}{n}\right)}{n\phi\left(h\right)}} \right). \tag{7.1}$$

Proof For $\chi \in S_{\mathcal{F}}$, let $K_i(\chi) = K_h(d(\chi, \chi_i))$ $(i = 1, ..., n)$. Based on the assumptions (A2)-(A4). if $K(1) > 0$ then by lemma 4.4 in [12]; or $K(1) > 0$ then by the boundedness of kernel $K(.)$, we have $c_1\phi(h) \leq \mathbb{E}[K_1(\chi)] \leq c_2\phi(h)$. Hence, in order to complete the proof, let us write

$g_n(\chi) = \frac{1}{n\mathbb{E}[K_1(\chi)]} \sum_{i=1}^{n} K_h(d(\chi_i, \chi)) \frac{\Delta_i B_i}{G(Y_i)}$ and $k(\chi) = \arg\min_{k \in 1,2,...,N_\varepsilon(S_{\mathcal{F}})} d(\chi, \chi_k)$, then it follows that:

$$\begin{aligned}
&\sup_{\chi \in S_{\mathcal{F}}} |g_n(\chi) - \mathbb{E}g_n(\chi)| \\
&\quad \leq \sup_{\chi \in S_{\mathcal{F}}} |g_n(\chi) - g_n(\chi_{k(\chi)})| + \sup_{\chi \in S_{\mathcal{F}}} |g_n(\chi_{k(\chi)}) - \mathbb{E}g_n(\chi_{k(\chi)})| \\
&+ \sup_{\chi \in S_{\mathcal{F}}} |\mathbb{E}g_n(\chi_{k(\chi)}) - \mathbb{E}g_n(\chi)| \\
&:= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned} \tag{7.2}$$

Let us first treat the term $\mathcal{I}_1$.

$$\begin{aligned}
\mathcal{I}_1 &= \sup_{\chi \in S_{\mathcal{F}}} \left| \sum_{i=1}^{n} \frac{1}{n\mathbb{E}[K_1(\chi)]} K_i(\chi) \frac{\Delta_i B_i}{G(Y_i)} - \frac{1}{n\mathbb{E}[K_1(\chi_{k(\chi)})]} K_i(\chi_{k(\chi)}) \frac{\Delta_i B_i}{G(Y_i)} \right| \\
&\leq \sup_{\chi \in S_{\mathcal{F}}} \frac{C}{n\phi(h)} \sum_{i=1}^{n} |K_i(\chi) - K_i(\chi_{k(\chi)})| I_{B(\chi,h) \cup B(\chi_{k(\chi)},h)}(\chi_i).
\end{aligned} \tag{7.3}$$

In the case of $K(1) = 0$ or $K(1) > 0$, similar to the proof of Lemma 8 in [10], by choosing $\varepsilon = n^{-1} \log n$, we get

$$\mathcal{I}_1 = O_p\left(\sqrt{\frac{\psi_{S_{\mathcal{F}}}\left(\frac{\log n}{n}\right)}{n\phi(h)}} \right). \tag{7.4}$$

Similarly,

$$\mathcal{I}_3 = O_p\left(\sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\log n}{n}\right)}{n\phi(h)}}\right). \tag{7.5}$$

In what follows, let us treat $\mathcal{I}_2$. Note that for all $\eta > 0$
$P\left(\mathcal{I}_2 > \eta\sqrt{\frac{\psi_{S_{\mathcal{F}}}(\varepsilon)}{n\phi\,(h)}}\right) = P\left(\max_{k\in\{1,\ldots,N_\varepsilon(S_{\mathcal{F}})\}} |g_n(\chi_k) - \mathbb{E}g_n(\chi_k)| > \eta\sqrt{\frac{\psi_{S_{\mathcal{F}}}(\varepsilon)}{n\phi\,(h)}}\right)$
$\leq N_\varepsilon(S_{\mathcal{F}}) \max_{k\in\{1,\ldots,N_\varepsilon(S_{\mathcal{F}})\}} P\left(|g_n(\chi_k) - \mathbb{E}g_n(\chi_k)| > \eta\sqrt{\frac{\psi_{S_{\mathcal{F}}}(\varepsilon)}{n\phi\,(h)}}\right)$. Writing $\Gamma_{ki} =:$ $\frac{1}{\mathbb{E}[K_1(\chi_k)]}\left\{\frac{\Delta_i B_i}{G(Y_i)}K_i(\chi_k) - \mathbb{E}\left(\frac{\Delta_i B_i}{G(Y_i)}K_i(\chi_k)\right)\right\}$, then it follows that $E|\Gamma_{ki}|^2 =$ $O(\phi(h)^{-1})$ by the same proof procedure as that of (6.27) in [12]. Thus, similar to the proof of F_2 of Lemma 8 in [10], it follows that

$$\mathcal{I}_2 = O_p\left(\sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\log n}{n}\right)}{n\phi(h)}}\right). \tag{7.6}$$

Combining (7.4)–(7.6) leads to

$$\sup_{\chi\in S_{\mathscr{F}}} |g_n(\chi) - \mathbb{E}g_n(\chi)| = O_p\left(\sqrt{\frac{\psi_{S_{\mathcal{F}}}\left(\frac{\log n}{n}\right)}{n\phi(h)}}\right). \tag{7.7}$$

Hence,

$$\begin{aligned}
&\sup_{\chi\in S_{\mathscr{F}}}\left|\frac{1}{n\phi\,(h)}\sum_{i=1}^{n}\left[K_h\left(d\left(\chi_i,\chi\right)\right)\frac{\Delta_i B_i}{G(Y_i)} - \mathbb{E}K_h\left(\chi_i,\chi\right)\frac{\Delta_i B_i}{G(Y_i)}\right]\right| \\
&\leq C.\sup_{x\in S_{\mathscr{F}}} |g_n(\chi) - \mathbb{E}g_n(\chi)| \\
&= O_p\left(\sqrt{\frac{\psi_{S_{\mathcal{F}}}\left(\frac{\log n}{n}\right)}{n\phi(h)}}\right),
\end{aligned} \tag{7.8}$$

then, the proof of Lemma 7.1 is completed.

In what follows, let $r_i = I(\varepsilon_{i\tau} \leq 0) - \tau$, $r_{ij} = I(\varepsilon_{i\tau} \leq \zeta_{ij}) - \tau$, where $\zeta_{ij} =$

$m^*_\tau(\chi_j) - m^*_\tau(\chi_i)$. Denoting $\theta_j = \sqrt{n\phi(h)}\big(\alpha_\tau(\chi_j) - m^*_\tau(\chi_j), (\beta_\tau - \beta^*_\tau)^\top\big)^\top$, $\widetilde{\theta}_j = \sqrt{n\phi(h)}\big(\widetilde{a}_\tau(\chi_j) - m^*_\tau(\chi_j), (\widetilde{\beta}_\tau - \beta^*_\tau)^\top\big)^\top$, $\mathbf{Z}^*_\mathbf{i} = \big(\mathbf{1}, \mathbf{Z}_\mathbf{i}^\top\big)^\top$, $\mathbf{Z}^* = \big(\mathbf{1}, \mathbf{Z}^\top\big)^\top$ and $\eta_{ij} = \frac{\mathbf{Z}_\mathbf{i}^{*\top}\theta_\mathbf{j}}{\sqrt{n\phi(h)}}$. Furthermore, for convenience, we quote the following identity of [21] in preparation for the latter technical proofs:

$$\rho_\tau(x-y) - \rho_\tau(x) = y\{(I(x \le 0) - \tau)\} + \int_0^y [I(x \le t) - I(x \le 0)]dt. \quad (7.9)$$

Lemma 7.2 *Suppose that the assumptions* (*A*1)-(*A*9)*hold, then we have*

$$\widetilde{\theta}_j = -\Big\{\mathbb{E} f_\tau(0|\mathbf{Z}, \boldsymbol{\chi}) K_h(d(\chi_j, \chi))\mathbf{Z}^*\mathbf{Z}^{*\top}\Big\}^{-1} \sqrt{\frac{\phi(h)}{n}} \sum_{i=1}^n \frac{\Delta_i}{G(Y_i)} \mathbf{Z}^*_i r_{ij} K_h(d(\chi_i, \chi_j)) + o_p(1). \quad (7.10)$$

Proof For $\chi \in S_\mathcal{F}$, noticing that

$$\widetilde{Y}_i - \mathbf{Z}_\mathbf{i}^\top \beta_\tau - \alpha_\tau(\chi_\mathbf{j}) = \varepsilon_{\mathbf{i}\tau} - \eta_\mathbf{ij} - \zeta_\mathbf{ij}. \quad (7.11)$$

Then, $\widetilde{\theta}_j$ is also the minimizer of

$$l^*(\theta_j) = \frac{1}{n}\sum_{i=1}^n \frac{\Delta_i}{\widehat{G}_n(Y_i)} \left\{\rho_\tau(\varepsilon_{i\tau} - \eta_{ij} - \zeta_{ij}) - \rho_\tau(\varepsilon_{i\tau} - \zeta_{ij})\right\} K_h(d(\chi_i, \chi_j)). \quad (7.12)$$

Similarly, let us denote

$$l(\theta_j) = \frac{1}{n}\sum_{i=1}^n \frac{\Delta_i}{G(Y_i)} \left\{\rho_\tau(\varepsilon_{i\tau} - \eta_{ij} - \zeta_{ij}) - \rho_\tau(\varepsilon_{i\tau} - \zeta_{ij})\right\} K_h(d(\chi_i, \chi_j)). \quad (7.13)$$

Thus, by the Taylor expansion [33], we have

$$\sqrt{n}\Big(\frac{1}{\widehat{G}_n(Y_i)} - \frac{1}{G(Y_i)}\Big) = \frac{1}{G(Y_i)}\frac{1}{\sqrt{n}}\sum_{j=1}^n \int_0^L I(Y_i \ge t)\frac{dM_j^C(t)}{y(t)} + o_p(1), \quad (7.14)$$

where $y(t) = \lim_{n\to\infty}(1/n)\sum_{i=1}^n I(Y_i \ge t)$ and $M_i^C(u) = (1-\delta_i)I(Y_i \le u) - \int_0^u I(Y_i \ge s)d\Lambda_C(s)$, respectively. Then, through the simple derivation, we can see that

$$\begin{aligned} & l^*(\theta_j) - l(\theta_j) \\ &= \frac{1}{n}\sum_{i=1}^n \left[\frac{\Delta_i}{\widehat{G}_n(Y_i)} - \frac{\Delta_i}{G(Y_i)}\right]\left\{\rho_\tau(\varepsilon_{i\tau} - \eta_{ij} - \zeta_{ij}) - \rho_\tau(\varepsilon_{i\tau} - \zeta_{ij})\right\} K_h(d(\chi_i, \chi_j)) \end{aligned}$$

$$= o_p(1) + \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\Delta_i}{G(Y_i)}\frac{1}{n}\sum_{j=1}^{n}\int_0^L I(Y_i \geq u)\frac{dM_j^C(u)}{y(u)}\right]$$
$$\left\{\rho_\tau(\varepsilon_{i\tau} - \eta_{ij} - \zeta_{ij}) - \rho_\tau(\varepsilon_{i\tau} - \zeta_{ij})\right\} K_h(d(\chi_i, \chi_j))$$
$$= o_p(1) + \frac{1}{n}\sum_{j=1}^{n}\int_0^L \left\{\frac{1}{n}\sum_{i=1}^{n}\frac{\Delta_i}{G(Y_i)}K_h(d(\chi_i, \chi_j))\frac{I(Y_i \geq u)}{y(u)}\right.$$
$$\left.\left[\rho_\tau(\varepsilon_{i\tau} - \eta_{ij} - \zeta_{ij}) - \rho_\tau(\varepsilon_{i\tau} - \zeta_{ij})\right]\right\} dM_j^C(u)$$
$$= o_p(1) + \frac{1}{n}\sum_{j=1}^{n}\int_0^L \left\{\frac{1}{n}\sum_{i=1}^{n}\frac{\Delta_i}{G(Y_i)}K_h(d(\chi_i, \chi_j))\frac{I(Y_i \geq u)}{y(u)}\right.$$
$$\left.\left[\int_0^{\eta_{ij}}(I(\varepsilon_{i\tau} \leq \zeta_{ij} + t) - \tau)dt\right]\right\} dM_j^C(u), \tag{7.15}$$

where the second identity is derived by incorporating (7.14), the third equation is derived by changing summation order, and the last equation is obatined from (7.9). Applying Lemma 7.1, the integrand in (7.15) has

$$\frac{1}{n\phi(h)}\sum_{i=1}^{n}\frac{\Delta_i}{G(Y_i)}K_h(d(\chi_i, \chi_j))\frac{I(Y_i \geq u)}{y(u)}\left[\int_0^{\eta_{ij}}(I(\varepsilon_{i\tau} \leq \zeta_{ij} + t) - \tau)dt\right]$$
$$= \frac{1}{\phi(h)}\mathbb{E}\left\{\frac{\Delta_i}{G(Y_i)}K_h(d(\chi_i, \chi_j))\frac{I(Y_i \geq u)}{y(u)}\left[\int_0^{\eta_{ij}}(I(\varepsilon_{i\tau} \leq \zeta_{ij} + t) - \tau)dt\right]\right\}$$
$$+ O_p\left(\sqrt{\frac{\psi_{S_\mathcal{F}}\left(\frac{\log n}{n}\right)}{n\phi(h)}}\right)$$
$$= \frac{1}{\phi(h)}\mathbb{E}\left\{\frac{\Delta_i}{G(Y_i)}K_h(d(\chi_i, \chi_j))\frac{I(Y_i \geq u)}{y(u)}\right.$$
$$\left.\left[\int_0^{\eta_{ij}}(F_\tau(\zeta_{ij} + t|\mathbf{Z}, \boldsymbol{\chi}) - F_\tau(0|Z_i, \chi_i))dt\right]\right\} \tag{7.16}$$
$$+ O_p\left(\sqrt{\frac{\psi_{S_\mathcal{F}}\left(\frac{\log n}{n}\right)}{n\phi(h)}}\right)$$
$$= \frac{1}{n(\phi(h))^2}\theta_j^T\mathbb{E}\left\{\frac{\Delta_i}{G(Y_i)}K_h(d(\chi_i, \chi_j))\frac{I(Y_i \geq u)}{y(u)}f_\tau(0|Z_i, \chi_i)Z_i^* Z_i^{*T}\right\}\theta_j$$
$$+ O_p\left(\sqrt{\frac{\psi_{S_\mathcal{F}}\left(\frac{\log n}{n}\right)}{n\phi(h)}}\right).$$

Under Assumption (A5) together with the compact set $S_{\mathcal{F}}$, combining (7.15) and (7.16) yields

$$\begin{aligned}
&l^*(\theta_j) - l(\theta_j) \\
&= o_p(1) + \frac{1}{n}\sum_{j=1}^{n}\int_0^L \left\{\frac{1}{n\phi(h)}\theta_j^T \mathbb{E}\right. \\
&\left.\left\{\frac{\Delta_i}{G(Y_i)}K_h(d(\chi_i,\chi_j))\frac{I(Y_i \geq u)}{y(u)}f_\tau(0|Z_i,\chi_i)Z_i^* Z_i^{*T}\right\}\theta_j\right\} dM_j^C(u). \quad (7.17)
\end{aligned}$$

Note that by assumptions (A1)-(A3) and (A6) and the properties of the conditional expectation again, one can get

$$\mathbb{E}\left\{\frac{\Delta_i}{G(Y_i)}K_h(d(\chi_i,\chi_j))\frac{I(Y_i \geq u)}{y(u)}f_\tau(0|Z_i,\chi_i)Z_i^* Z_i^{*T}\right\} = O_p(1).$$

Thus, by the matingale central limit theorems, it then follows from (7.17) that

$$|l^*(\theta_j) - l(\theta_j)| = o_p(1). \quad (7.18)$$

Furthermore, by (7.9) and (7.13), we have

$$\begin{aligned}
l(\theta_j) &= \frac{1}{n}\sum_{i=1}^{n}\frac{\Delta_i}{G(Y_i)}\left\{\rho_\tau(\varepsilon_{i\tau} - \eta_{ij} - \zeta_{ij}) - \rho_\tau(\varepsilon_{i\tau} - \zeta_{ij})\right\} K_h(d(\chi_i,\chi_j)) \\
&:= \frac{1}{n}\sum_{i=1}^{n}\frac{\Delta_i}{G(Y_i)}\eta_{ij} r_{ij} K_h(d(\chi_i,\chi_j)) + \frac{1}{n}R_n(\theta_j). \quad (7.19)
\end{aligned}$$

Then, by (A1),(A7) and a simple arithmetic, It follows that

$$\begin{aligned}
\mathbb{E}(R_n(\theta_j)|\mathbf{Z},\boldsymbol{\chi}) &= \sum_{i=1}^{n} K_h(d(\chi_i,\chi_j))\int_0^{\eta_{ij}} (F_\tau(\zeta_{ij} + t|\mathbf{Z},\boldsymbol{\chi}) - F_\tau(\zeta_{ij}|\mathbf{Z},\boldsymbol{\chi}))dt \\
&:= \frac{1}{2}\theta_j^\top Q_j \theta_j + O_p(h^\alpha), \quad (7.20)
\end{aligned}$$

where $Q_j = \sum_{i=1}^{n} \frac{1}{n\phi(h)} f_\tau(0|\mathbf{Z},\boldsymbol{\chi})K_h(d(\chi_i,\chi_j))\mathbf{Z}_i^*\mathbf{Z}_i^{*\top}$. Hence, by [34], we also have $Var(R_n(\theta_j)|\mathbf{Z},\boldsymbol{\chi}) = o_p(1)$, which leads to

$$R_n(\theta_j) = \mathbb{E}(R_n(\theta_j)|\mathbf{Z},\boldsymbol{\chi}) + o_p(1). \quad (7.21)$$

On the other hand, by Lemma 7.1 and (7.20), we have

$$Q_j = \mathbb{E}(Q_j) + O_p\left(\sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\log n}{n}\right)}{n\phi(h)}}\right) \tag{7.22}$$

and

$$R_n(\theta_j) = \frac{1}{2}\theta_j^\top \mathbb{E}(Q_j)\theta_j + O_p\left(h^\alpha + \sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{logn}{n}\right)}{n\phi(h)}}\right), \tag{7.23}$$

respectively. Hence, by (7.19)–(7.23), it follows that

$$\begin{aligned} l(\theta_j) &= \frac{1}{n}\sum_{i=1}^{n}\frac{\Delta_i}{G(Y_i)}\eta_{ij}r_{ij}K_h(d(\chi_i,\chi_j)) \\ &+ \frac{1}{2n}\theta_j^\top\frac{1}{\phi(h)}\mathbb{E}\left\{f_\tau(0|\mathbf{Z},\boldsymbol{\chi})K_h(d(\chi_i,\chi_j))\mathbf{Z}^*\mathbf{Z}^{*\top}\right\}\theta_j \\ &+ O_p\left(\frac{1}{n}\sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\log n}{n}\right)}{n\phi(h)}} + \frac{h^\alpha}{n}\right). \end{aligned}$$

Then, following the ideas of Theorems 5.7 and 5.23 in Chapter 5 of [36], the minimizer of $l^*(\theta_j)$ can be written as

$$\begin{aligned} \widetilde{\theta}_j &= -\left\{\frac{1}{\phi(h)}\mathbb{E}f_\tau(0|\mathbf{Z},\boldsymbol{\chi})K_h(d(\chi,\chi_j))\mathbf{Z}^*\mathbf{Z}^{*\top}\right\}^{-1} \\ &\sum_{i=1}^{n}\frac{\Delta_i}{\widehat{G}_n(Y_i)}\frac{\mathbf{Z}^*}{\sqrt{n\phi(h)}}r_{ij}K_h(d(\chi_i,\chi_j)) \\ &+ O_p\left(\frac{1}{n}\sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\log n}{n}\right)}{n\phi(h)}} + \frac{h^\alpha}{n}\right). \end{aligned} \tag{7.24}$$

Thus, the proof of Lemma 7.2 is completed.

Proof of Theorem 3.1 Similar to [8], let us re-define $\widehat{\theta} = \sqrt{n}(\widehat{\beta}_\tau - \beta^*_\tau)$ and $\zeta_i = \widetilde{a}_\tau(\chi_i) - m^*_\tau(\chi_i)$, respectively. Then, $\widehat{\theta}$ is also the minimizer of the following function:

$$\begin{aligned}
l^*(\theta) &= \frac{1}{n}\sum_{i=1}^{n}\frac{\Delta_i}{\widehat{G}_n(Y_i)}\left\{\rho_\tau\left(\varepsilon_{i\tau} - \zeta_i - Z_i^\top\frac{\theta}{\sqrt{n}}\right) - \rho_\tau\left(\varepsilon_{i\tau} - \zeta_i\right)\right\} \\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{\Delta_i}{G(Y_i)}\left\{\rho_\tau\left(\varepsilon_{i\tau} - \zeta_i - Z_i^\top\frac{\theta}{\sqrt{n}}\right) - \rho_\tau\left(\varepsilon_{i\tau} - \zeta_i\right)\right\} \\
&+ \frac{1}{n}\sum_{i=1}^{n}\frac{\Delta_i(G(Y_i) - \widehat{G}_n(Y_i))}{\widehat{G}_n(Y_i)G(Y_i)}\left[\rho_\tau\left(\varepsilon_{i\tau} - \zeta_i - Z_i^\top\frac{\theta}{\sqrt{n}}\right) - \rho_\tau(\varepsilon_{i\tau} - \zeta_i)\right] \\
&:= \frac{1}{n}R_{1n}(\theta) + \frac{1}{n}R_{2n}(\theta). \qquad (7.25)
\end{aligned}$$

By (7.9), it follows that

$$\begin{aligned}
R_{1n}(\theta) &= \sum_{i=1}^{n}\frac{\Delta_i}{G(Y_i)}\frac{\mathbf{Z_i}^\top\theta}{\sqrt{n}}r_i + \sum_{i=1}^{n}\frac{\Delta_i}{G(Y_i)}\int_{\zeta_i}^{\zeta_i+\mathbf{Z_i}^\top\theta/\sqrt{n}}[I(\varepsilon_{i\tau} \le t) - I(\varepsilon_{i\tau} \le 0)]dt \\
&=: \sum_{i=1}^{n}\frac{\Delta_i}{G(Y_i)}\frac{\mathbf{Z_i}^\top\theta}{\sqrt{n}}r_i + S_{1n}(\theta). \qquad (7.26)
\end{aligned}$$

Hence, by the conditional independence given $\mathbf{Z}_i$, χ_i and the fact that $\mathbb{E}\left(\frac{\Delta_i}{G(Y_i)}|\mathbf{Z}_i, \chi_i\right) = 1$ a.s, it follows that

$$\begin{aligned}
&\mathbb{E}(S_{1n}(\theta)) \\
&= \sum_{i=1}^{n}\mathbb{E}\left\{\mathbb{E}\left[\frac{\Delta_i}{G(Y_i)}\int_{\zeta_i}^{\zeta_i+\mathbf{Z_i}^\top\theta/\sqrt{n}}\big(I(\varepsilon_{i\tau} \le t) - I(\varepsilon_{i\tau} \le 0)\big)dt|\mathbf{Z}_i, \chi_i\right]\right\} \\
&= \sum_{i=1}^{n}\mathbb{E}\int_{\zeta_i}^{\zeta_i+\mathbf{Z_i}^\top\theta/\sqrt{n}}\big(F_\tau(t|\mathbf{Z}, \chi) - F_\tau(0|\mathbf{Z}, \chi)\big)dt \\
&= \sum_{i=1}^{n}\frac{1}{2n}f_\tau(0|\mathbf{Z}, \boldsymbol{\chi})\cdot\theta^\top\mathbf{Z_i}\mathbf{Z_i}^\top\theta + \sum_{i=1}^{n}\frac{\mathbf{Z_i}^\top\theta}{\sqrt{n}}f_\tau(0|\mathbf{Z}, \boldsymbol{\chi})\zeta_i + o(1). \qquad (7.27)
\end{aligned}$$

By Lemma 7.2, we have that

$$\begin{aligned}
&\sum_{i=1}^{n}\frac{\mathbf{Z}_i}{\sqrt{n}}f_\tau(0|\mathbf{Z}, \boldsymbol{\chi})\zeta_i \\
&\quad = -\frac{1}{\sqrt{n}}\sum_{i=1}^{n}f_\tau(0|\mathbf{Z}, \boldsymbol{\chi})\cdot\mathbf{Z}_i(1, \mathbf{0}^\top)\cdot \\
&\quad\quad \frac{1}{n}\left(\mathbb{E}\{K_h(d(\chi_i, \chi))\mathbb{E}[f_\tau(0|\mathbf{Z}, \boldsymbol{\chi})\mathbf{Z}^*\mathbf{Z}^{*\top}|\boldsymbol{\chi}]\}\right)^{-1} \times
\end{aligned}$$

$$\sum_{j=1}^{n} \mathbf{Z}_j^* r_j K_h(d(\chi_i, \chi_j)) \frac{\Delta_j}{G(Y_j)}$$
$$+ O_p\left(\frac{1}{n\phi(h)} \sqrt{\frac{\psi_{S_{\mathcal{F}}}\left(\frac{\log n}{n}\right)}{n}} + \frac{h^\alpha}{n\sqrt{\phi(h)}} \right)$$
$$= -\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \frac{\Delta_j}{G(Y_j)} r_j A_\tau(\chi_j, Z_j) + o_p(1). \tag{7.28}$$

Performing similar calculations as those in (7.21), one can get $S_{1n}(\theta) = \mathbb{E}(S_{1n}(\theta)) + o_p(1)$. Then, by (7.26)-(7.28), we have that

$$R_{1n}(\theta) = \frac{1}{2}\theta^\top \left(\sum_{i=1}^{n} \frac{1}{n} f_\tau(0|\mathbf{Z}, \boldsymbol{\chi}) Z_i Z_i^\top \right) \theta$$
$$+ \left\{ \sum_{i=1}^{n} \frac{\Delta_i}{G(Y_i)} (Z_i - A_\tau(\chi_i, z_i)) \frac{r_i}{\sqrt{n}} \right\}^\top \theta + o_p(1) \tag{7.29}$$

and

$$R_{2n}(\theta) = \sum_{i=1}^{n} \frac{\Delta_i (G(Y_i) - \widehat{G}_n(Y_i))}{\widehat{G}_n(Y_i) G(Y_i)} \left[\rho_\tau(\varepsilon_{i\tau} - \zeta_i - \mathbf{Z}_i^\top \theta/\sqrt{n}) - \rho_\tau(\varepsilon_{i\tau} - \zeta_i) \right]$$
$$= \sum_{i=1}^{n} \frac{(G(Y_i) - \widehat{G}_n(Y_i))}{\widehat{G}_n(Y_i)} \left[\left\{ \frac{\Delta_i}{G(Y_i)} (\mathbf{Z}_i - A_\tau(\chi_i, z_i)) \frac{r_i}{\sqrt{n}} \right\}^\top \theta \right] + o_p(1), \tag{7.30}$$

respectively, where the last equality of (7.30) is derived from (7.14) and

$$\left| \sum_{i=1}^{n} \frac{(G(Y_i) - \widehat{G}_n(Y_i))}{\widehat{G}_n(Y_i)} \theta^\top \frac{1}{n} f_\tau(0|\mathbf{Z}, \boldsymbol{\chi}) \mathbf{Z}_i \mathbf{Z}_i^\top \theta \right|$$
$$= \frac{1}{n} \sum_{j=1}^{n} \int_0^L \theta^\top \left[\frac{1}{n} \sum_{i=1}^{n} I(Y_i \geq u) f_\tau(0|\mathbf{Z}, \boldsymbol{\chi}) \mathbf{Z}_i \mathbf{Z}_i^\top \right] \theta / y(u) dM_j^C(u) = o_p(1).$$

In what follows, we further denote

$$T_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\Delta_i}{G(Y_i)} \left[1 + \frac{G(Y_i) - \widehat{G}_n(Y_i)}{\widehat{G}_n(Y_i)} \right] \big(\mathbf{Z}_i - A_\tau(\chi_i, Z_i) \big) r_i,$$

which leads to

$$l^*(\theta) = \frac{1}{2}\theta^\top \left(\sum_{i=1}^{n} \frac{1}{n} f_\tau(0|\mathbf{Z}, \boldsymbol{\chi}) \mathbf{Z}_i \mathbf{Z}_i^\top \right) \theta + T_n^\top \theta + o_p(1). \tag{7.31}$$

Similar to [8, 26] and [4], the minimizer of $l^*(\theta)$ is

$$\widehat{\theta} = -\left(\sum_{i=1}^{n} \frac{1}{n} f(0|\mathbf{Z}, \boldsymbol{\chi}) \mathbf{Z}_i \mathbf{Z}_i^\top \right)^{-1} T_n + o_p(1). \tag{7.32}$$

By (7.14) and the facts that

$$\frac{\Delta_i}{G(Y_i)} = 1 - \int_0^L \frac{dM_i^C(u)}{G(u)},$$

we obtain

$$\begin{aligned} T_n = & \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[(\mathbf{Z}_i - A_\tau(\chi_i, Z_i)) r_i - \int_0^L \frac{dM_i^C(u)}{G(u)} (\mathbf{Z}_i - A_\tau(\chi_i, Z_i)) r_i \right] \\ & + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\Delta_i}{G(Y_i)} \frac{1}{n} \sum_{j=1}^{n} \int_0^L I(Y_i \geq u)(\mathbf{Z}_i - A_\tau(\chi_i, Z_i)) r_i \frac{dM_j^C(u)}{y(u)} + o_p(1). \end{aligned}$$

Then it follows that

$$\begin{aligned} T_n = & \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Bigg[(\mathbf{Z}_i - A_\tau(\chi_i, Z_i)) r_i - \int_0^L (\mathbf{Z}_i - A_\tau(\chi_i, Z_i)) r_i \\ & - \frac{\sum_{j=1}^{n} \frac{\Delta_j}{G(Y_j)} \frac{1}{n} (I(Y_j \geq u)(\mathbf{Z}_j - A_\tau(\chi_j, Z_j))) r_j}{S(u)} \frac{dM_i^C(u)}{G(u)} \Bigg] \\ & + o_p(1), \end{aligned} \tag{7.33}$$

where $S(u) = P(\widetilde{Y} \geq u)$. According to the martingale central limit theorem, we have

$$\sqrt{n}\left(\hat{\beta}_\tau - \beta_\tau\right) \xrightarrow{d} N\left(0, \Sigma_{1\tau}^{-1} \Sigma_{2\tau} \Sigma_{1\tau}^{-1}\right), \tag{7.34}$$

where $\Sigma_{1\tau} = \mathbb{E}\left(f_\tau(0|\mathbf{Z}, \boldsymbol{\chi}) \mathbf{Z}\mathbf{Z}^\top\right)$ and

$$\Sigma_{2\tau} = Var(T_n) = \mathbb{E}\left[((\mathbf{Z} - A_\tau(\chi, z)) r)^{\otimes 2}\right]$$

$$+\mathbb{E}\left[\left(\int_0^\tau (\mathbf{Z}-A_\tau(\chi,z))r-\frac{\sum_{j=1}^n \frac{\Delta_j}{G(Y_j)}\frac{1}{n}(I(Y_j\geq u)(\mathbf{Z}_j-A_\tau(\chi_j,z_j)))r_j}{S(u)}\right)^{\otimes 2}\right.$$
$$\left. I(Y\geq u)\frac{\lambda(u)}{G^2(u)}du\right].$$

□

Proof of Theorem 3.2 For $\chi\in S_{\mathcal{F}}$, we also re-define $\widehat{\theta}=\sqrt{n\phi(h)}(\widehat{a}_\tau(\chi)-m_\tau^*(\chi))$ and $\zeta_i=m_\tau^*(\chi)-m_\tau^*(\chi_i)$ respectively. Similar to the decomposition of (7.11), we have that

$$\begin{aligned}Y_i-\mathbf{Z}_i^\top\widehat{\beta}_\tau-\hat{a}_\tau(\chi)\\ &=\varepsilon_{i\tau}-\mathbf{Z}_i(\widehat{\beta}_\tau-\beta_\tau^*)-(\widehat{a}_\tau(\chi)-m_\tau^*(\chi))-(m_\tau^*(\chi)-m_\tau^*(\chi_i))\\ &=\varepsilon_{i\tau}-\mathbf{Z}_i^\top(\widehat{\beta}_\tau-\beta_\tau)-\zeta_i-\frac{1}{\sqrt{n\phi(h)}}\widehat{\theta}. \qquad (7.35)\end{aligned}$$

By the proof procedure of Lemma 7.2, let

$$l^*(\widehat{\theta})=\frac{1}{n}\sum_{i=1}^n\frac{\Delta_i}{\widehat{G}_n(Y_i)}\left\{\rho_\tau\left(\varepsilon_{i\tau}-Z_i^\top(\widehat{\beta}_\tau-\beta_\tau)-\zeta_i-\frac{\widehat{\theta}}{\sqrt{n\phi(h)}}\right)\right.$$
$$\left.-\rho_\tau\left(\varepsilon_{i\tau}-Z_i^\top(\widehat{\beta}_\tau-\beta_\tau)-\zeta_i\right)\right\}K_h(d(\chi_i,\chi)) \qquad (7.36)$$

and

$$l(\widehat{\theta})=\frac{1}{n}\sum_{i=1}^n\frac{\Delta_i}{G(Y_i)}\left\{\rho_\tau\left(\varepsilon_{i\tau}-Z_i^\top(\widehat{\beta}_\tau-\beta_\tau)-\zeta_i-\frac{\widehat{\theta}}{\sqrt{n\phi(h)}}\right)\right.$$
$$\left.-\rho_\tau\left(\varepsilon_{i\tau}-Z_i^\top(\widehat{\beta}_\tau-\beta_\tau)-\zeta_i\right)\right\}K_h(d(\chi_i,\chi)). \qquad (7.37)$$

Performing the same calculations as those of (7.15)–(7.17), we can also get $|l^*(\widehat{\theta})-l(\widehat{\theta})|=o_p(1)$

In what follows, for simplicity, we denote $s_i=I(\varepsilon_{i\tau}-Z_i(\widehat{\beta}_\tau-\beta_\tau)-\zeta_i<0)-\tau$. By the Knight identity (7.9), $l(\widehat{\theta})$ can be written as

$$\begin{aligned}l(\widehat{\theta})&=\frac{1}{n}\sum_{i=1}^n\frac{\Delta_i}{G(Y_i)}K_h(d(\chi_i,\chi))\frac{\widehat{\theta}}{\sqrt{n\phi(h)}}\cdot s_i\\ &+\sum_{i=1}^n\frac{\Delta_i}{G(Y_i)}\int_0^{\frac{\widehat{\theta}}{\sqrt{n\phi(h)}}}[I(\varepsilon_{i\tau}-Z_i^\top(\widehat{\beta}_\tau-\beta_\tau)-\zeta_i<t)\\ &-I(\varepsilon_{i\tau}-Z_i^\top(\widehat{\beta}_\tau-\beta_\tau)-\zeta_i<0))dt]K_h(d(\chi_i,\chi))\end{aligned}$$

$$:= \frac{1}{n} W_n \widehat{\theta} + \frac{1}{n} B_n(\widehat{\theta}). \tag{7.38}$$

Similar to the proof procedure of (7.27), it follows that

$$\mathbb{E}(B_n(\widehat{\theta})) = \sum_{i=1}^{n} \mathbb{E}\left[\int_0^{\frac{\widehat{\theta}}{\sqrt{n\phi(h)}}} \{I(\varepsilon_{i\tau} - Z_i^\top(\widehat{\beta_\tau} - \beta_\tau) - \zeta_i < t) \right.$$
$$\left. - I(\varepsilon_{i\tau} - Z_i^\top(\widehat{\beta_\tau} - \beta_\tau) - \zeta_i < 0)\} dt \cdot K_h(d(\chi_i, \chi)) \right]$$
$$= \frac{1}{2} \sum_{i=1}^{n} \frac{\widehat{\theta}^\top}{n\phi(h)} f_\tau(0|\mathbf{Z}, \boldsymbol{\chi}) \cdot K_h(d(\chi_i, \chi)) \widehat{\theta} + o(1).$$

Next, let $Q^* = \frac{1}{\phi(h)} \mathbb{E}[K_h(d(\boldsymbol{\chi}, \chi)) \cdot \mathbb{E}(f_\tau(0|\mathbf{Z}, \boldsymbol{\chi})|\chi)]$. Following the same proof procedure as that of Lemma 7.2 and Theorem 3.1, we obtain

$$\widehat{\theta} = -Q^{*-1} \cdot W_n + O_p\left(\frac{1}{n} \sqrt{\frac{\psi_{S_\mathscr{F}}\left(\frac{\log n}{n}\right)}{n\phi(h)}} + \frac{h^\alpha}{n} \right). \tag{7.39}$$

Furthermore, let us denote

$$W_{1n} = \frac{1}{n\phi(h)} \sum_{i=1}^{n} \frac{\Delta_i}{G(Y_i)} \cdot K_h(d(\chi_i, \chi)) \cdot r_i,$$
$$W_{2n} = \frac{1}{n\phi(h)} \sum_{i=1}^{n} \frac{\Delta_i}{G(Y_i)} K_h(d(\chi_i, \chi))(s_i - r_i)$$

and $Q_1 = -\mathbb{E}[f_\tau(0|\mathbf{Z}, \boldsymbol{\chi})|\boldsymbol{\chi} = \chi]$. Then, we obtain $Q^* = O_p(1) Q_1$, which leads to

$$\sup_{\chi \in S_\mathscr{F}} |\widehat{a}_\tau(\chi) - m_\tau(\chi)| = O_p(1) \sup_{\chi \in S_\mathscr{F}} \left| Q_1^{-1}(W_{1n} + W_{2n}) \right|$$
$$+ O_p\left(\frac{\sqrt{\psi_{S_\mathscr{F}}\left(\frac{\log n}{n}\right)}}{n^2\phi(h)} + \frac{h^\alpha}{n\sqrt{n\phi(h)}} \right).$$

To check the upper bound of $|\widehat{a}_\tau(\chi) - m_\tau(\chi)|$ precisely, let us treat term W_{2n}. By Taylor's expansion, it follows that

$$\begin{aligned}\mathbb{E}(W_{2n}|\mathbf{Z}, \boldsymbol{\chi}) &= \mathbb{E}\left\{\frac{1}{n\phi(h)}\sum_{i=1}^{n}\frac{\Delta_i}{G(Y_i)}K_h(d(\chi_i, \chi))(s_i - r_i)|\mathbf{Z}, \boldsymbol{\chi}\right\} \\ &= O_p(h^\alpha) + O_p\left(\frac{1}{\sqrt{n}}\right) + o_p(h^\alpha) = O_p(h^\alpha).\end{aligned}$$

Similarly, we have $Var(W_{2n}|\mathbf{Z}, \boldsymbol{\chi}) = o_p(h^\alpha)$. Thus, by Chebyschev's inequality, it follows that

$$W_{2n} = O_p(h^\alpha).$$

Next, by Lemma 7.1, we obtain

$$\sup_{x\in S_{\mathscr{F}}} |W_{1n}| = \mathbb{E}W_{1n} + O_p\left(\sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\log n}{n}\right)}{n\phi(h)}}\right) = O_p\left(\sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\log n}{n}\right)}{n\phi(h)}}\right).$$

Finally, by assumption (A7), we obtain that

$$\begin{aligned}\sup_{\chi\in S_{\mathscr{F}}} |\widehat{a}_\tau(\chi) - m_\tau(\chi)| &= O_p\left(h^\alpha + \sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\log n}{n}\right)}{n\phi(h)}} + \frac{\sqrt{\psi_{S_{\mathscr{F}}}\left(\frac{\log n}{n}\right)}}{n^2\phi(h)} + \frac{h^\alpha}{n\sqrt{n\phi(h)}}\right) \\ &= O_p\left(h^\alpha + \sqrt{\frac{\psi_{S_{\mathscr{F}}}\left(\frac{\log n}{n}\right)}{n\phi(h)}}\right).\end{aligned}$$

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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